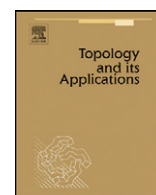


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## Topology and its Applications

[www.elsevier.com/locate/topol](http://www.elsevier.com/locate/topol)On real moduli spaces of holomorphic bundles over  $M$ -curves<sup>☆</sup>Nikolai Saveliev<sup>a,\*</sup>, Shuguang Wang<sup>b</sup><sup>a</sup> Department of Mathematics, University of Miami, PO Box 249085, Coral Gables, FL 33124, United States<sup>b</sup> Department of Mathematics, University of Missouri, 202 Mathematical Sciences Bldg, Columbia, MO 65211, United States

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## ABSTRACT

Let  $F$  be a genus  $g$  curve and  $\sigma : F \rightarrow F$  a real structure with the maximal possible number of fixed circles. We study the real moduli space  $\mathcal{N}' = \text{Fix}(\sigma^\#)$  where  $\sigma^\# : \mathcal{N} \rightarrow \mathcal{N}$  is the induced real structure on the moduli space  $\mathcal{N}$  of stable holomorphic bundles of rank 2 over  $F$  with fixed non-trivial determinant. In particular, we calculate  $H^*(\mathcal{N}', \mathbb{Z})$  in the case of  $g = 2$ , generalizing Thaddeus' approach to computing  $H^*(\mathcal{N}, \mathbb{Z})$ .

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## 0. Introduction

In their influential paper [1], Atiyah and Bott used two-dimensional Yang–Mills theory to compute cohomology of the moduli space  $\mathcal{N}$  of stable holomorphic bundles over a Riemann surface  $F$ . In essence, the computation was inspired by the idea that the Yang–Mills functional in this dimension should be an equivariantly perfect Morse–Bott function (this was proved later in full generality by Daskalopoulos [2]).

A real structure  $\sigma : F \rightarrow F$  induces a real structure  $\sigma^\#$  on the moduli space  $\mathcal{N}$ . Understanding the structure of the real moduli space  $\mathcal{N}' = \text{Fix}(\sigma^\#)$  is an important problem which has proved to be elusive. Only recently was it addressed in part in a series of papers by N.-K. Ho and C.-C. M. Li, the latest being [6]. They mainly treated the case when the real structure  $\sigma$  has no fixed points, which led them to the study of the Yang–Mills functional on the non-orientable surface  $F/\sigma$ .

In this paper we consider the other extreme, when  $\sigma$  has the maximal possible number of fixed circles. In this case, the pair  $(F, \sigma)$  is usually referred to as an  $M$ -curve [12]. More specifically, we work with the moduli space  $\mathcal{N}$  of stable holomorphic bundles of rank 2 over  $F$  with fixed non-trivial determinant, and with the associated real moduli space  $\mathcal{N}'$ . Instead of the infinite dimensional Yang–Mills functional as in [1,6], we utilize a finite dimensional Morse–Bott function (defined in Section 10)

$$f' : \mathcal{N}' \rightarrow \mathbb{R}. \quad (1)$$

In the context of the complex moduli space  $\mathcal{N}$ , a similar function  $f : \mathcal{N} \rightarrow \mathbb{R}$  was first introduced by Jeffrey and Weitsman [7]. Later Thaddeus [9] proved directly that the function  $f$  is perfect and thus recalculated the cohomology of  $\mathcal{N}$ .

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For technical reasons, in this paper we will only take up the case of genus two  $M$ -curves, hoping to address the case of higher genera elsewhere. For  $M$ -curves of genus two, the second author [10] described  $\mathcal{N}'$  algebraically as the intersection of two quadrics in the five-dimensional real projective space; however, it turned out to be rather difficult to extract any further information about the topology of  $\mathcal{N}'$  from that description. The Morse theoretic approach of this paper, on the other hand, gives a complete calculation of the integral cohomology of  $\mathcal{N}'$ .

**Theorem 1.** *Let  $(F, \sigma)$  be a genus two  $M$ -curve and  $\mathcal{N}'$  the real moduli space of stable holomorphic bundles of rank 2 over  $F$  with fixed non-trivial determinant. Then the Morse–Bott function (1) is perfect over  $\mathbb{Z}$  and, at the level of graded abelian groups, there is an isomorphism  $H^*(\mathcal{N}', \mathbb{Z}) = H^*(S^1 \times S^1 \times S^1, \mathbb{Z})$ .*

It should be mentioned that the above isomorphism does not hold at the level of cohomology rings, see Remark 9.1. For higher genera, we prove the following partial result.

**Theorem 2.** *Let  $(F, \sigma)$  be an  $M$ -curve of genus  $g$ . Then the real moduli space of stable holomorphic bundles of rank 2 over  $F$  with fixed non-trivial determinant is connected.*

We also propose the following conjecture.

**Conjecture.** *Let  $(F, \sigma)$  be an  $M$ -curve of genus  $g \geq 1$ , and  $\mathcal{N}'_g$  the real moduli space of stable holomorphic bundles of rank 2 over  $F$  with fixed non-trivial determinant. Then the Morse–Bott function (1) is perfect over  $\mathbb{Z}/2$ .*

Unlike Theorem 1 which holds over  $\mathbb{Z}$ , the above conjecture is only stated over  $\mathbb{Z}/2$ . The reason for that is that, while the perfection of  $f : \mathcal{N} \rightarrow \mathbb{R}$  in Thaddeus' paper [9] was suggested by the work of Frankel [4], the perfection of  $f'$  is suggested in part by Duistermaat's paper [3], which only states the tightness theorem over  $\mathbb{Z}/2$ .

By working out the critical submanifolds of  $f'$  together with their Morse–Bott indices, we conclude the following.

**Proposition 3.** *Assuming that the above conjecture holds true, the Poincaré polynomial of  $H^*(\mathcal{N}'_g, \mathbb{Z}_2)$  is given by*

$$P_t(\mathcal{N}'_g) = (1+t)^{g-1} \cdot \frac{(1+t^2)^g - (2t)^g}{(1-t)^2}. \quad (2)$$

In particular, it follows that  $\mathcal{N}'_g$  is an  $M$ -variety in  $\mathcal{N}_g$  in that

$$\sum_k \dim H^k(\mathcal{N}'_g, \mathbb{Z}/2) = \sum_l \dim H^l(\mathcal{N}_g, \mathbb{Z}).$$

Formula (2) would be highly desirable, since practically nothing is known about the real moduli space  $\mathcal{N}'_g$ , other than the results of [6] for free real structures. Note that (2) holds for  $g = 2$  by Theorem 1, and it certainly holds for  $g = 1$  because the real moduli space is then a single point.

Parts of this project were accomplished while we attended the 2006 Summer Session of the Park City/IAS Mathematics Institute. We express our appreciation to the organizers for providing a stimulating environment.

## 1. Real moduli spaces over curves

Let  $F$  be a compact surface of genus  $g \geq 2$  and fix a point  $p \in F$ . Denote by  $\mathcal{N}$  the moduli space of stable holomorphic bundles  $\mathcal{E} \rightarrow F$  of rank 2 with determinant  $\mathcal{L}_p^{-1}$ , where  $p \in F$  is viewed as a divisor. Then  $\mathcal{N}$  is a smooth complex manifold of real dimension  $6g - 6$ , modeled on the complex vector space  $H^1(F, \text{End } \mathcal{E})$  by the deformation theory.

Let  $\sigma : F \rightarrow F$  be a real structure on  $F$  whose real part  $F' = \text{Fix}(\sigma)$  is non-empty. Then  $F'$  contains at least one circle, but may contain as many as  $g + 1$  circles (Harnack's inequality, see for instance [12]). In the latter case, the pair  $(F, \sigma)$  is called an  $M$ -curve. Choose  $p \in F'$  then  $\sigma : F \rightarrow F$  induces an involution  $\sigma^\# : \mathcal{N} \rightarrow \mathcal{N}$  by the formula

$$\sigma^\#[\mathcal{E}] = [\sigma^* \bar{\mathcal{E}}],$$

where  $\bar{\mathcal{E}}$  stands for the complex conjugate of  $\mathcal{E}$ . Note that since  $\sigma : F \rightarrow F$  is orientation reversing,  $\sigma^* \mathcal{L}_p = \bar{\mathcal{L}}_p$ , thus making the complex conjugation in the above formula necessary.

Note that the linear map  $H^1(F, \text{End } \mathcal{E}) \rightarrow H^1(F, \text{End}(\sigma^* \bar{\mathcal{E}}))$  induced by  $\sigma$  is a complex conjugation, therefore, the involution  $\sigma^\# : \mathcal{N} \rightarrow \mathcal{N}$  is anti-holomorphic and hence is a real structure on  $\mathcal{N}$ . It follows that the real moduli space  $\mathcal{N}' = \text{Fix}(\sigma^\#)$  is a closed smooth manifold of dimension  $3g - 3$ .

**Proposition 1.1.** *The manifold  $\mathcal{N}'$  is orientable.*

**Proof.** According to [1], the moduli space  $\mathcal{N}$  is simply connected and spin. Therefore, the real structure  $\sigma^\#$  must be compatible with the unique spin structure on  $\mathcal{N}$  in the sense of [11]. The main result in [11] then applies to infer that  $\mathcal{N}'$  is orientable. Note that  $\sigma^\#$  need not preserve the spin structure in the usual sense, since it is orientation reversing when the complex dimension of  $\mathcal{N}$  is odd, that is, when  $g$  is even.  $\square$

## 2. Representation varieties

Let  $F_0$  be the surface  $F$  punctured at  $p \in F$ . The theorem of Narasimhan and Seshadri [8] can be used to identify  $\mathcal{N}$  with the representation variety  $\mathcal{M}$  which consists of the conjugacy classes of  $SU(2)$  representations of  $\pi_1(F_0) = \pi_1(F_0, x_0)$  sending  $[\partial F_0] \in \pi_1(F_0)$  to  $-1 \in SU(2)$  (the latter condition does not depend on the choice of basepoint because  $-1$  is a central element in  $SU(2)$ ).

Given a real structure  $\sigma : F \rightarrow F$  with non-empty  $F' = \text{Fix}(\sigma)$  and  $p \in F'$  as above, choose a basepoint  $x_0 \in F' \cap F_0$ . Then we have an induced involution  $\sigma_* : \pi_1(F_0) \rightarrow \pi_1(F_0)$ , which in turn induces an involution  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$  by the formula

$$\sigma^*[\alpha] = [\alpha \circ \sigma_*].$$

That  $\sigma^*$  is well defined follows from the fact that  $\sigma_*[\partial F_0] = [\partial F_0]^{-1} = -1$ .

**Lemma 2.1.** *Let  $\varphi : \mathcal{M} \rightarrow \mathcal{N}$  be the Narasimhan–Seshadri diffeomorphism then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\sigma^*} & \mathcal{M} \\ \varphi \downarrow & & \downarrow \varphi \\ \mathcal{N} & \xrightarrow{\sigma^\#} & \mathcal{N} \end{array}$$

**Proof.** The Narasimhan–Seshadri correspondence assigns to every  $[\alpha] \in \mathcal{M}$  a stable holomorphic bundle  $\mathcal{E}_\alpha \rightarrow F$  as follows. Let  $\tilde{F}_0 \rightarrow F_0$  be the universal covering space of  $F_0$  and consider the holomorphic bundle  $\mathcal{E} \rightarrow F_0$  with

$$\mathcal{E} = \tilde{F}_0 \times_{\pi_1(F_0)} \mathbb{C}^2,$$

where  $\pi_1(F_0)$  acts on  $\mathbb{C}^2$  via  $\alpha : \pi_1(F_0) \rightarrow SU(2)$ . Obviously,  $\det \mathcal{E}$  is trivial. Since the holonomy of  $\alpha$  along the loop  $\partial F_0$  is fixed, we can trivialize  $\mathcal{E}$  near the boundary  $\partial F_0$ . Glue  $D^2 \times \mathbb{C}^2$  in using the transition function  $z^{-1}$  along  $\partial F_0$ . The result is the stable bundle  $\mathcal{E}_\alpha \rightarrow F$  with the determinant  $\det \mathcal{E}_\alpha = \mathcal{L}_p^{-1}$  yielding the Narasimhan–Seshadri correspondence.

For any given  $[\alpha] \in \mathcal{M}$  we have  $\sigma^\#[\mathcal{E}_\alpha] = [\sigma^* \overline{\mathcal{E}_\alpha}] = [\mathcal{E}_{\sigma^* \alpha}]$ , where  $\overline{\alpha}$  is the complex conjugate of  $\alpha$ . However, for any matrix

$$A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2),$$

its complex conjugate

$$\bar{A} = \begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

is the matrix conjugate of  $A$  via

$$j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SU(2)$$

(we use the standard identification between  $SU(2)$  matrices and unit quaternions). This means that  $\bar{\alpha}$  and  $\alpha$  are conjugate representations, and therefore the bundles  $\mathcal{E}_{\sigma^* \alpha}$  and  $\mathcal{E}_{\sigma^* \alpha}$  are isomorphic.  $\square$

Denote by  $\mathcal{M}'$  the fixed point set of the involution  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$  then the above lemma implies that  $\mathcal{M}'$  and  $\mathcal{N}'$  are diffeomorphic.

**Corollary 2.2.** *The moduli space  $\mathcal{M}'$  is a smooth closed orientable manifold of dimension  $3g - 3$ .*

In conclusion, note that  $\mathcal{M}$  is a symplectic manifold with symplectic form  $\omega : H^1(F; \text{ad } \rho) \otimes H^1(F; \text{ad } \rho) \rightarrow \mathbb{R}$  given by  $\omega(u, v) = -1/2 \text{tr}(u \cup v)[F]$ , see Goldman [5].

**Lemma 2.3.** *The map  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$  is a real structure with respect to the symplectic form  $\omega$ , that is,  $\sigma^* \omega = -\omega$ .*

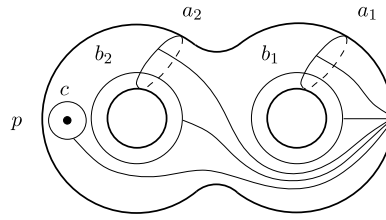


Fig. 1.

**Proof.** This result is obtained by the following straightforward calculation:

$$\begin{aligned} (\sigma^* \omega)(u, v) &= \omega(\sigma^* u, \sigma^* v) = -1/2 \operatorname{tr}(\sigma^* u \cup \sigma^* v)[F] \\ &= -1/2 \operatorname{tr}(\sigma^*(u \cup v))[F] = -1/2 \operatorname{tr}(u \cup v) \sigma_*[F] = -\omega(u, v). \end{aligned}$$

In the last equality, we used the fact that the map  $\sigma : F \rightarrow F$  is orientation reversing.  $\square$

### 3. The case of $g = 2$

Let  $(F, \sigma)$  be a genus two  $M$ -curve embedded in  $\mathbb{R}^3$  as shown in Fig. 1, with  $\sigma : F \rightarrow F$  acting as the reflection in the horizontal plane fixing the obvious three circles. Let  $F_0$  be a once punctured surface  $F$ , the puncture  $p \in F$  being a real point on the left circle of  $F'$  (the other two options for positioning  $p$  will be treated in Section 9). Then  $\pi_1(F_0)$  is the free group

$$\pi_1(F_0) = \langle a_1, b_1, a_2, b_2 \mid \rangle$$

whose generators  $a_1, a_2, b_1, b_2$  are shown in Fig. 1. Also shown is the curve  $c = [a_1, b_1] \cdot [a_2, b_2]$ . All of the above curves are oriented clockwise with respect to the projection shown in Fig. 1.

The moduli space  $\mathcal{M}$  can now be described as follows. Let us consider the smooth map  $\mu : SU(2)^4 \rightarrow SU(2)$  given by  $\mu(A_1, B_1, A_2, B_2) = -[A_1, B_1] \cdot [A_2, B_2]$ . Since  $1 \in SU(2)$  is a regular value of  $\mu$ , its preimage  $\mu^{-1}(1)$  is a smooth manifold. It contains no reducibles, that is, 4-tuples  $(A_1, B_1, A_2, B_2)$  whose entries commute with each other, because the latter would contradict the equation  $[A_1, B_1] \cdot [A_2, B_2] = -1$ . Therefore, the action of  $SO(3) = SU(2)/\pm 1$  on  $\mu^{-1}(1)$  by conjugation is free, and its quotient space is then the smooth six-dimensional manifold  $\mathcal{M}$ . A straightforward calculation shows that the induced action  $\sigma : \pi_1(F_0) \rightarrow \pi_1(F_0)$  is given on the generators by the formulas

$$\begin{aligned} \sigma_*(a_1) &= a_1, & \sigma_*(b_1) &= b_1^{-1}, \\ \sigma_*(a_2) &= b_1^{-1} c b_2 a_2 b_2^{-1} b_1, & \sigma_*(b_2) &= b_1^{-1} b_2^{-1} b_1. \end{aligned}$$

In practical terms,  $\mathcal{M}$  consists of the conjugacy classes  $[A_1, B_1, A_2, B_2]$  of quadruples  $(A_1, B_1, A_2, B_2)$  such that  $[A_1, B_1] \cdot [A_2, B_2] = -1$ , and the real structure  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$  is given by

$$\sigma^*[A_1, B_1, A_2, B_2] = [B_1 A_1 B_1^{-1}, B_1^{-1}, -B_2 A_2 B_2^{-1}, B_2^{-1}].$$

As we already know, the fixed point set  $\mathcal{M}'$  of  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$  is a smooth orientable manifold of dimension 3.

### 4. The function $f'$

Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be the function on the moduli space  $\mathcal{M}$  defined by the formula

$$f([A_1, B_1, A_2, B_2]) = \operatorname{tr}(B_1)/2.$$

This is a smooth function whose range is  $[-1, 1]$ , and  $f^{-1}(-1, 1)$  is acted upon by  $S^1$  as follows. After conjugation if necessary, every element of  $f^{-1}(-1, 1)$  can be written in the form  $[A_1, B_1, A_2, B_2]$  with  $B_1 = e^{i\beta}$  and  $0 < \beta < \pi$  (we continue to identify  $SU(2)$  with the group of unit quaternions). The circle action is then given by  $e^{i\varphi} : [A_1, B_1, A_2, B_2] \mapsto [A_1 e^{i\varphi}, B_1, A_2, B_2]$ .

**Lemma 4.1.** *This is a well-defined action on  $f^{-1}(-1, 1)$ .*

**Proof.** Any other choice of representative in  $[A_1, B_1, A_2, B_2]$  with  $B_1 = e^{i\gamma}$  and  $0 < \gamma < \pi$  will have the property that  $e^{i\gamma} = g e^{i\beta} g^{-1}$  for some  $g \in SU(2)$ . Therefore,  $\gamma = \beta$  and  $g$  is a unit complex number. Because of that,  $(g A_1 g^{-1}) e^{i\varphi} = g (A_1 e^{i\varphi}) g^{-1}$ , making the action well defined. Of course, if  $B_1 = \pm 1$ , the conjugating element  $g$  can be any  $SU(2)$  matrix, and the above argument fails.  $\square$

According to Goldman [5], the above circle action preserves the symplectic form  $\omega$  on  $\mathcal{M}$  and  $\arccos f$  is its moment map (up to a factor of  $i$ ).

It is clear that  $f \circ \sigma^* = f$  hence  $f : \mathcal{M} \rightarrow \mathbb{R}$  is invariant with respect to  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$ . The restriction of  $f$  to  $\mathcal{M}'$  will be denoted by  $f' : \mathcal{M}' \rightarrow \mathbb{R}$ . The range of  $f'$  is again  $[-1, 1]$ . The circle action on  $f^{-1}(-1, 1)$  is not defined on  $(f')^{-1}(-1, 1)$ . Nevertheless, it gives rise to the residual  $\mathbb{Z}/2$  action  $r : \mathcal{M}' \rightarrow \mathcal{M}'$  defined on the entire moduli space  $\mathcal{M}'$  (and in fact on the full moduli space  $\mathcal{M}$ ) by the formula

$$r([A_1, B_1, A_2, B_2]) = [-A_1, B_1, A_2, B_2]. \quad (3)$$

## 5. The critical submanifolds of $f'$

Thaddeus [9, Lemma 3.3] proved that  $f : \mathcal{M} \rightarrow \mathbb{R}$  is a Morse–Bott function and described its critical submanifolds. One can easily check that the critical submanifolds of  $f$  intersect  $\mathcal{M}'$  transversely, and that these intersections are precisely the critical submanifolds of  $f' : \mathcal{M}' \rightarrow \mathbb{R}$ . They are of two types.

The first type is comprised of the manifolds  $S'_1, S'_3 \subset \mathcal{M}'$  on which  $f'$  achieves its absolute minimum and maximum. To be precise, the absolute minimum  $S_1 = f^{-1}(-1)$  of  $f : \mathcal{M} \rightarrow \mathbb{R}$  is a copy of  $SU(2)$  consisting of quadruples  $[A_1, -1, i, j]$  parametrized by  $A_1 \in SU(2)$ . The action

$$\sigma^*[A_1, -1, i, j] = [A_1, -1, i, -j] = [iA_1i^{-1}, -1, i, j]$$

corresponds to the map  $\text{Ad}_i : SU(2) \rightarrow SU(2)$  sending  $A_1$  to  $iA_1i^{-1}$ . The fixed point set of this action, which is  $S'_1$ , is the circle consisting of quadruples  $[e^{i\varphi}, -1, i, j]$  parametrized by  $e^{i\varphi}$ . Similarly,  $S_3 = f^{-1}(1)$ , where  $f : \mathcal{M} \rightarrow \mathbb{R}$  achieves its absolute maximum, gives rise to the circle  $S'_3$  consisting of quadruples  $[e^{i\psi}, 1, i, j]$ .

**Lemma 5.1.** *The critical submanifolds  $S'_1$  and  $S'_3$  of  $f'$  are non-degenerate in the Morse–Bott sense, and their indices are respectively 0 and 2.*

**Proof.** The Hessian of  $f$  is negative definite on the normal bundle of  $S_3 \subset \mathcal{M}$ . But then its restriction to the normal bundle of  $S'_3 \subset \mathcal{M}'$ , which is the Hessian of  $f'$ , is also negative definite. In particular,  $S'_3$  is non-degenerate, and its Morse–Bott index equals the codimension of  $S'_3$  in  $\mathcal{M}'$ , which is 2. The argument for  $S'_1$  is analogous.  $\square$

Within  $f^{-1}(-1, 1)$ , the critical points of  $f : \mathcal{M} \rightarrow \mathbb{R}$  coincide with those of the moment map  $\arccos f$ , hence with the fixed points of the circle action on  $f^{-1}(-1, 1)$ . They form the submanifold  $S_2 \subset \mathcal{M}$  which is a 2-torus consisting of quadruples  $[j, i, z, w]$  parametrized by  $z, w \in \mathbb{C}$  such that  $|z| = |w| = 1$ . Note that  $f = 0$  on  $S_2$ . One can easily see that the fixed point set  $S'_2$  of the involution  $\sigma^* : S_2 \rightarrow S_2$  given by

$$\sigma^*[j, i, z, w] = [-j, -i, -z, \bar{w}] = [j, i, -\bar{z}, w]$$

consists of the two circles  $[j, i, \pm i, w]$  with  $|w| = 1$ .

**Lemma 5.2.** *The critical submanifold  $S'_2$  of  $f'$  is non-degenerate in the Morse–Bott sense, and its index is equal to 1.*

**Proof.** According to Frankel [4], the moment map  $\arccos f$  is a Morse–Bott function on  $f^{-1}(-1, 1)$ , and hence so is the function  $f$ . The involution  $\mathcal{M} \rightarrow \mathcal{M}$  given by  $[A_1, B_1, A_2, B_2] \mapsto [A_1, -B_1, A_2, B_2]$  changes sign of  $f$ . Since  $S_2$  is connected, the index of  $f$  is half the rank of the normal bundle of  $S_2 \subset \mathcal{M}$ , or 2. The involution  $\sigma^*$  is antisymplectic, see Lemma 2.3, hence we can apply [3, Proposition 2.2] to the moment map  $\arccos f$ . It tells us that  $\arccos f'$  and hence  $f'$  are Morse–Bott on  $S'_2$ , and the index of  $f'$  is half that of  $f$ .  $\square$

**Lemma 5.3.** *The residual involution  $r : \mathcal{M}' \rightarrow \mathcal{M}'$  acts as the  $180^\circ$  rotation on each of the circles  $S'_1$  and  $S'_3$ , and acts trivially on  $S'_2$ .*

**Proof.** The circle  $S'_3$  consists of the quadruples  $[e^{i\psi}, 1, i, j]$  acted upon by  $r$  as  $[e^{i\psi}, 1, i, j] \rightarrow [-e^{i\psi}, 1, i, j]$ . The case of  $S'_1$  is completely similar. The manifold  $S'_2$  consists of the quadruples  $[j, i, \pm i, w]$  where  $w$  is a unit complex number. The action of  $r$  is given by  $[j, i, \pm i, w] \rightarrow [-j, i, \pm i, w] = [j, i, \pm i, w]$ , after conjugating by  $i$ .  $\square$

## 6. The Morse–Bott spectral sequence

As we have seen, the critical set of the Morse–Bott function  $f' : \mathcal{M}' \rightarrow \mathbb{R}$  is a disjoint union  $S'_1 \cup S'_2 \cup S'_3$ , where the index of  $S'_p$  is equal to  $p - 1$ , and the restriction of  $f'$  to each of the  $S'_p$  is constant,  $f'(S'_p) = p - 2$  for  $p = 1, 2, 3$ . Let  $x_j = j - 3/2$  for  $j = 0, 1, 2, 3$ , and consider the filtration

$$\emptyset = U'_0 \subset U'_1 \subset U'_2 \subset U'_3 = \mathcal{M}'$$

$q$					
	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0	
	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0	
					$p$

Fig. 2.

of  $\mathcal{M}'$  by the open sets  $U'_j = (f')^{-1}(x_0, x_j)$ . The Morse Lemma implies that, up to homotopy,  $U'_j$  is a complex obtained from  $U'_{j-1}$  by attaching, along its boundary, the disc bundle associated to the negative normal bundle over  $S'_j$  whose fibers are the negative definite subspaces of the Hessian of  $f'$  on  $S'_j$ . The  $E_1$  term of the Morse–Bott spectral sequence associated with this filtration is of the form shown in Fig. 2, where the  $(p-1)$ st column represents  $H^*(S'_p)$  for  $p = 1, 2, 3$ . This spectral sequence converges to  $H^*(\mathcal{M}')$ .

Observe that the filtration is preserved by the residual  $\mathbb{Z}/2$  action  $r : \mathcal{M}' \rightarrow \mathcal{M}'$  hence  $r$  induces an automorphism  $r^*$  of the above spectral sequence. Our next task will be to compute the differentials, of which only  $d_1$  and  $d_2$  are not obviously zero.

## 7. Differential $d_1$

The differential  $d_1 : E_1^{0,0} \rightarrow E_1^{1,0}$  must vanish because  $H^0(\mathcal{M}') \neq 0$ . In particular, we conclude that  $H^0(\mathcal{M}') = \mathbb{Z}$  and hence  $\mathcal{M}'$  is connected. Similarly, the differential  $d_1 : E_1^{1,1} \rightarrow E_1^{2,1}$  must vanish because  $\mathcal{M}'$  is orientable, see Corollary 2.2, so  $H^3(\mathcal{M}') = \mathbb{Z}$ . In fact, we will show that the remaining differentials  $d_1$  also vanish but this will take some effort.

Let us first consider the differential  $d_1 : E_1^{1,0} \rightarrow E_1^{2,0}$ . It is represented as the composition

$$H^0(S'_2) \rightarrow H^1(U'_2, U'_1) \rightarrow H^1(U'_2) \rightarrow H^2(U'_3, U'_2) \rightarrow H^0(S'_3). \quad (4)$$

Here, the second and third arrows are portions of the long exact sequences of the respective pairs hence they commute with the automorphism  $r^*$ . The first and the last arrows are Thom isomorphisms, therefore, their behavior with respect to  $r^*$  is determined by how  $r$  acts on  $S'_2$  and  $S'_3$  and on their respective negative normal bundles. According to Lemma 5.3, the action of  $r^*$  on both  $H^0(S'_2)$  and  $H^0(S'_3)$  is trivial.

**Lemma 7.1.** *The involution  $r$  acts as minus identity on the normal bundle of  $S'_2 \subset \mathcal{M}'$  and, in particular, on its negative normal bundle.*

**Proof.** This will follow from the fact that  $r : \mathcal{M} \rightarrow \mathcal{M}$  given by the formula (3) is an involution and that its fixed point set coincides with  $S_2$ . To show the latter, suppose that  $[A_1, B_1, A_2, B_2] = [-A_1, B_1, A_2, B_2]$ . Then there is  $u \in SU(2)$  such that  $uA_1 = -A_1u$  and  $u$  commutes with  $B_1, A_2$  and  $B_2$ . Since  $(A_1, B_1, A_2, B_2)$  is an irreducible representation, we conclude that  $u^2 = 1$  or  $u^2 = -1$ . The former cannot happen because then  $-A_1 = A_1$  and  $A_1 = 0$ , a contradiction with  $A_1 \in SU(2)$ . Therefore, we have  $u^2 = -1$ , and  $u = \pm i$  after conjugation. The fact that  $u = \pm i$  commutes with  $B_1, A_2$  and  $B_2$  means that all three of them are unit complex numbers. In fact, one can conjugate by  $j$  if necessary to make  $B_1 = e^{i\beta}$  with  $\beta \in [0, \pi]$ , perhaps at the expense of changing the sign of  $u$ . Conjugate further by a unit complex number to make  $A_1 = a + bj$  with real non-negative  $b$ , while preserving  $u, B_1, A_2$  and  $B_2$ . The relation  $[A_1, B_1] = [A_1, B_1] \cdot [A_2, B_2] = -1$  then tells us that, up to conjugation,  $A_1 = j$  and  $B_1 = i$ . Therefore, the only fixed points of  $r : \mathcal{M} \rightarrow \mathcal{M}$  are of the form  $[j, i, z, w]$ , with  $z$  and  $w$  some unit complex numbers. This is exactly  $S_2 \subset \mathcal{M}$ .  $\square$

Since the rank of the negative normal bundle to  $S'_2$  is one,  $r$  reverses orientation on the fiber. Therefore,  $r^*$  anticommutes with the first arrow in (4). On the other hand,  $S'_3$  is the absolute maximum hence its negative normal bundle is the same as its normal bundle. According to Lemma 5.3, the action of  $r$  on  $S'_3$  is orientation preserving. Since  $r : \mathcal{M}' \rightarrow \mathcal{M}'$  preserves orientation (which follows for example from Lemma 7.1) it must be orientation preserving on the negative normal bundle of  $S'_3$ . Therefore,  $r^*$  commutes with the last arrow in (4). In conclusion,  $d_1 : E_1^{1,0} \rightarrow E_1^{2,0}$  anticommutes with  $r^*$  and thus must vanish.

Finally, let us consider the differential  $d_1 : E_1^{0,1} \rightarrow E_1^{1,1}$ . It is represented as the composition

$$H^1(S'_1) \rightarrow H^1(U'_1, U'_0) \rightarrow H^1(U'_1) \rightarrow H^2(U'_2, U'_1) \rightarrow H^1(S'_2). \quad (5)$$

As before, the two arrows in the middle commute with  $r^*$ , and the first and the last arrows are Thom isomorphisms. The action of  $r$  on  $S'_1$  is by the  $180^\circ$  rotation hence the induced action on  $H^1(S'_1)$  is trivial. Since the negative normal bundle of  $S'_1$  is zero-dimensional, there is no Thom class to worry about and we readily conclude that  $r^*$  commutes with the first arrow in (5). Concerning the last arrow, we already know that  $r^*$  anticommutes with the Thom class of the negative normal

bundle of  $S'_2$ . On the other hand,  $r$  acts as the identity on  $S'_2$  and hence as the identity on  $H^1(S'_2)$ . This implies that  $r^*$  anticommutes with the last arrow and hence with the composition (5). Thus  $d_1 : E_1^{0,1} \rightarrow E_1^{1,1}$  vanishes.

## 8. Differential $d_2$

The results of the previous section imply that  $E_2 = E_1$  hence all that remains to do to prove that  $f'$  is perfect is to show that the differential  $d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$  vanishes. We will show that the edge homomorphism  $i^* : H^1(\mathcal{M}') \rightarrow H^1(S'_1)$  in the spectral sequence induced by the inclusion  $i : S'_1 \rightarrow \mathcal{M}'$  is surjective. This will imply that  $d_2 = 0$  because a generator of  $E_2^{0,1} = H^1(S'_1) = \mathbb{Z}$  must survive in the  $E_\infty$  term of the spectral sequence, hence it must be in the kernel of  $d_2$ . Note that a similar argument could have been used to show vanishing of  $d_1 : E_1^{0,1} \rightarrow E_1^{1,1}$  above.

Remember that  $S'_1$  consists of the quadruples  $[e^{i\varphi}, -1, i, j] \in \mathcal{M}'$ . Consider the subset  $\mathcal{R}' \subset \mathcal{M}'$  which consists of quadruples  $[1, B_1, A_2, B_2]$  fixed by the involution  $\sigma^* : \mathcal{M}' \rightarrow \mathcal{M}'$ .

**Lemma 8.1.**  $\mathcal{R}' \subset \mathcal{M}'$  is a smoothly embedded 2-sphere which intersects  $S'_1 \subset \mathcal{M}'$  transversely in exactly one point.

**Proof.** The relation  $[1, B_1] \cdot [A_2, B_2] = [A_2, B_2] = -1$  on the points of  $\mathcal{R}'$  implies that, up to conjugation,  $A_2 = i$  and  $B_2 = j$ . The fact that  $[1, B_1, i, j] \in \mathcal{R}'$  is fixed by  $\sigma^*$  means that  $\sigma^*([1, B_1, i, j]) = [1, B_1^{-1}, i, -j] = [1, iB_1^{-1}i^{-1}, i, j]$  (we used conjugation by  $i$  in the last equality). Therefore,  $\mathcal{R}'$  is parametrized by  $B_1 \in SU(2)$  such that  $B_1 i = iB_1^{-1}$ . These are precisely the unit quaternions with no  $i$  component; they obviously comprise an embedded 2-sphere in  $\mathcal{M}'$ . The intersection  $\mathcal{R}' \cap S'_1$  consists of just one point,  $[1, -1, i, j]$ , and it is obviously transversal.  $\square$

Now, the lemma implies that the natural generator of  $H^1(S'_1) = \mathbb{Z}$  is the image under  $i^*$  of the Poincaré dual of  $\mathcal{R}'$ . This shows that  $i^* : H^1(\mathcal{M}') \rightarrow H^1(S'_1)$  is surjective and thus completes the argument.

## 9. Proof of Theorem 1

The Morse–Bott spectral sequence associated with the function  $f' : \mathcal{M}' \rightarrow \mathbb{R}$  converges to  $H^*(\mathcal{M}') = H^*(\mathcal{N}')$ . As we showed in the last two sections, all the differentials in this spectral sequence vanish and therefore it collapses at the  $E_1$  term,  $E_1 = E_\infty$ . This completes the proof in the case when the puncture belongs to the left circle of  $F'$  as shown in Fig. 1.

If the puncture belongs to the right circle of  $F'$ , the involution  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$  is given by the formula

$$\sigma^*[A_1, B_1, A_2, B_2] = [-B_1 A_1 B_1^{-1}, B_1^{-1}, B_2 A_2 B_2^{-1}, B_2^{-1}],$$

and Theorem 1 follows by simply renaming the variables. In the remaining case, when  $p \in F'$  belongs to the middle circle, the involution  $\sigma^* : \mathcal{M} \rightarrow \mathcal{M}$  is given by

$$\sigma^*[A_1, B_1, A_2, B_2] = [B_1 A_1 B_1^{-1}, B_1^{-1}, B_2 A_2 B_2^{-1}, B_2^{-1}].$$

The above proof goes through with minimal changes, which can be safely left to the reader.

**Remark 9.1.** The regular neighborhood of  $S'_1 \cup \mathcal{R}' = S^1 \vee S^2$  in  $\mathcal{M}'$  is a punctured  $S^1 \times S^2$ . This implies that  $\mathcal{M}'$  splits into a connected sum, one of the factors being  $S^1 \times S^2$ . In particular, the isomorphism  $H^*(\mathcal{N}') = H^*(S^1 \times S^1 \times S^1)$  of Theorem 1 is not a ring isomorphism.

## 10. The higher genera

In this section, we will give a brief outline of the calculation that has led us to Conjecture 2 and Proposition 3. Let  $(F, \sigma)$  be an  $M$ -curve of genus  $g \geq 2$ , and  $\mathcal{M}_g$  the representation variety identified as in Section 2 with the moduli space  $\mathcal{N}_g$ . The map  $\sigma_* : \pi_1(F_0) \rightarrow \pi_1(F_0)$  is given on the standard generators of the fundamental group by the formula

$$\sigma_*(a_m) = (b_{m-1} \cdots b_1)^{-1} c_{m-1} a_m (b_{m-1} \cdots b_1),$$

$$\sigma_*(b_m) = (b_{m-1} \cdots b_1)^{-1} b_m^{-1} (b_{m-1} \cdots b_1),$$

where  $c_m = [a_1, b_1][a_2, b_2] \cdots [a_m, b_m]$  for all  $m = 1, \dots, n$ . It induces the real structure  $\sigma^* : \mathcal{M}_g \rightarrow \mathcal{M}_g$  with the fixed point set  $\mathcal{M}'_g$ .

Following [7] and [9], define a smooth function  $f : \mathcal{M}_g \rightarrow \mathbb{R}$  by the formula

$$f([A_1, B_1, \dots, A_g, B_g]) = \text{tr}(B_1)/2.$$

This is a Morse–Bott function which is invariant with respect to  $\sigma^*$ , hence it restricts to a function  $f' : \mathcal{M}'_g \rightarrow \mathbb{R}$ . The range of both  $f$  and  $f'$  is  $[-1, 1]$ .

The critical manifolds of  $f'$ , which are the fixed point sets of  $\sigma^*$  acting on the critical manifolds of  $f$ , are of two types. First, we have the absolute minimum  $S'_1 = (f')^{-1}(-1)$  and the absolute maximum  $S'_3 = (f')^{-1}(1)$ . An easy calculation shows that the projection

$$\pi([A_1, \pm 1, A_2, B_2, \dots, A_g, B_g]) = [A_2, B_2, \dots, A_g, B_g]$$

makes each of  $S'_1$  and  $S'_3$  into an  $S^1$ -bundle over  $\mathcal{M}'_{g-1}$ . These bundles admit a section given by  $A_1 = 1$ , and the Gysin sequence implies that  $H^*(S'_1, \mathbb{Z}) = H^*(S'_3, \mathbb{Z}) = H^*(S^1, \mathbb{Z}) \otimes H^*(\mathcal{M}'_{g-1}, \mathbb{Z})$ . Note that the Morse–Bott index of  $S'_1$  is zero, and the Morse–Bott index of  $S'_3$  is the codimension of  $S'_3$  in  $\mathcal{M}'_g$ , that is,  $\dim \mathcal{M}'_g - (\dim \mathcal{M}'_{g-1} + 1) = 2$ .

The remaining critical submanifold of  $f'$  is the fixed point set  $S'_2$  of  $\sigma^*$  acting on the critical manifold  $S_2$  of  $f$ . The latter, according to Thaddeus [9], is the  $(2g-2)$ -dimensional torus that consists of the  $2g$ -tuples  $[j, i, A_2, B_2, \dots, A_g, B_g]$ , where  $A_2, B_2, \dots, A_g, B_g$  are unit complex numbers. One can easily calculate that the action of  $\sigma^*$  on  $S_2$  is given by the formula

$$\sigma^*([j, i, A_2, B_2, \dots, A_g, B_g]) = [j, i, -A_2^{-1}, B_2, \dots, A_g^{-1}, B_g],$$

and that the fixed point set  $S'_2$  consists of  $2^{g-1}$  copies of the torus  $T^{g-1}$ . An argument similar to that of Lemma 5.2 shows that the Morse–Bott index of  $S'_2$  is half that of  $S_2$ , hence equals  $g-1$ .

The above calculation implies in particular that  $\mathcal{M}'$  is connected, thus proving Theorem 2.

As in Thaddeus [9], one may construct a symplectic cut  $\hat{\mathcal{M}}_g$  from  $\mathcal{M}_g$  to resolve the issue that  $S^1$  does not act globally on  $\mathcal{M}_g$  (although the residual  $\mathbb{Z}/2$  does act on  $\mathcal{M}'_g$ ). The  $S^1$  action on  $\hat{\mathcal{M}}_g$  is Hamiltonian and hence gives rise to a moment map  $\hat{f}$ , which is a perfect Morse–Bott function according to [4]. It is not difficult to check that  $\hat{\mathcal{M}}_g$  inherits a real structure  $\hat{\sigma}$  from  $\mathcal{M}_g$  and  $\hat{\sigma}$  is compatible with the  $S^1$ -action. Duistermaat's main theorem on tightness [3] then implies that the restriction of  $\hat{f}$  to the real part  $\hat{\mathcal{M}}'_g$  is perfect over  $\mathbb{Z}/2$ .

Combining the above perfection with the parity arguments similar to those for genus two case, it may be possible to show the vanishing of the differentials in the Morse–Bott spectral sequence of  $f' : \mathcal{M}'_g \rightarrow \mathbb{R}$ . This would imply the perfection of  $f'$  over  $\mathbb{Z}/2$ , as stated in our conjecture.

Assuming the conjecture, the above calculation with the Morse–Bott function  $f' : \mathcal{M}' \rightarrow \mathbb{R}$  leads to the recursive relation

$$P_t(\mathcal{M}'_g) = (1+t)(1+t^2)P_t(\mathcal{M}'_{g-1}) + 2^{g-1}(1+t)t^{g-1}t^{g-1}$$

on the Poincaré polynomial  $P_t(\mathcal{M}'_g)$  (with  $\mathbb{Z}/2$  coefficients). It is straightforward to see that this relation is satisfied by (2), proving the first part of Proposition 3. To show that  $\mathcal{M}'_g \subset \mathcal{M}_g$  is an  $M$ -variety amounts to showing that  $P_t(\mathcal{M}'_g) = P_t(\mathcal{M}_g)$  at  $t=1$ , where  $P_t(\mathcal{M}_g)$  is the Poincaré polynomial of  $\mathcal{M}_g$  over  $\mathbb{Z}$ . This is immediate using either the formula

$$P_t(\mathcal{M}_g) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}$$

or the recursive relation

$$P_t(\mathcal{M}_g) = (1+t^3)^2 P_t(\mathcal{M}_{g-1}) + t^{2g-2}(1+t)^{2g-2},$$

see for instance Thaddeus [9].

Note that by Smith theory, if  $\mathcal{M}'_g \subset \mathcal{M}_g$  is an  $M$ -variety then the real structure  $\sigma$  induces the trivial map on  $H^*(\mathcal{M}_g, \mathbb{Z}/2)$ , see for instance [12].

Finally, note that our approach does not work for free real structures  $\sigma$ , since we have to choose a real basepoint in Section 2. Other than  $M$ -curves, the next case to try might be that of dividing curves, where the real part  $F'$  divides  $F$  into two components. In general, we believe the function  $f'$  will no longer be perfect.

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